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Orthogonal Polynomial Solutions of Spectral Type Differential Equations: Magnus' Conjecture[1]

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Let $\tau=\sigma+\nu$ be a point mass perturbation of a classical moment functional σ by a distribution ν with finite support. We find necessary conditions for the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$, orthogonal relative to τ, to be a Bochner–Krall orthogonal polynomial system (*BKOPS*); that is, $\{Q_n(x)\}_{n=0}^{\infty}$ are eigenfunctions of a finite order linear differential operator of spectral type with polynomial coefficients: $L_N[y](x)=\sum_{i=1}^{N}\ell_i(x)\,y^{(i)}(x)=\lambda_n y(x)$. In particular, when ν is of order 0 as a distribution, we find necessary and sufficient conditions for $\{Q_n(x)\}_{n=0}^{\infty}$ to be a *BKOPS*, which strongly support and clarify Magnus' conjecture which states that any *BKOPS* must be orthogonal relative to a classical moment functional plus one or two point masses at the end point(s) of the interval of orthogonality. This result explains not only why the Bessel-type orthogonal polynomials (found by Hendriksen) cannot be a *BKOPS* but also explains the phenomena for infinite-order differential equations (found by J. Koekoek and R. Koekoek), which have the generalized Jacobi polynomials and the generalized Laguerre polynomials as eigenfunctions.

Key Words: differential equations; Bochner–Krall orthogonal polynomials; Magnus' conjecture.

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1. INTRODUCTION

In this work, we are interested in an orthogonal polynomial system (*OPS*) which satisfies a linear differential equation of spectral type,

$$(1.1)\qquad L_N[y](x)=\sum_{i=1}^{N}\ell_i(x)\,y^{(i)}(x)=\sum_{i=1}^{N}\sum_{j=0}^{i}\ell_{i,j}x^jy^{(i)}(x)=\lambda_n\,y(x),$$

where $\ell_{i,j}$ are real constants and $\lambda_n=\ell_{11}n+\cdots+\ell_{NN}n(n-1)\cdots(n-N+1)$.

In 1929, Bochner [3] showed that there are essentially (up to a complex linear change of variable) five polynomial sequences (namely, the four classical orthogonal polynomials of Jacobi, Bessel, Laguerre, and Hermite, and $\{x^n\}_{n=0}^{\infty}$) that satisfy the differential equation (1.1) with $N=2$. Kwon and Littlejohn [28] followed Bochner's work by showing that, up to a real change of variable, there are six distinct *OPS*'s (Jacobi, Bessel, Laguerre, Hermite, twisted Jacobi, and twisted Hermite polynomials) that arise as eigenfunctions of the differential equation (1.1) with $N=2$. Although Bochner [3] did not discuss the orthogonality of the polynomial sequences that he classified, the orthogonality of the Jacobi, Laguerre, and Hermite polynomials was clearly understood. The complex orthogonality of the Bessel polynomials was later observed and studied in detail by Krall and Frink [23] (see also [6, 25, 37]). Generalizing Bochner's classification problem, H. L. Krall [21] found necessary and sufficient conditions for an *OPS* to satisfy a differential equation (1.1) of arbitrary order, by which he also classified [22] (up to a complex linear change of variable) all *OPS*'s satisfying fourth order differential equations of the form (1.1). In addition to rediscovering four classical *OPS*'s of Jacobi, Bessel, Laguerre, Hermite, he also found three new *OPS*'s satisfying fourth order equations. A. M. Krall [19] called these three new *OPS*'s classical-type *OPS*'s since they are orthogonal relative to moment functionals which are point mass perturbations of classical moment functionals. Generalizing classical-type *OPS*'s, Koornwinder [18] introduced the generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$, which are orthogonal on $[-1,1]$ relative to the classical Jacobi weight plus two point masses at $x=\pm 1$, given explicitly by

$$w_j^{(\alpha,\beta,M,N)}(x)=\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}(\alpha+1)\,\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}$$
$$+M\delta(x+1)+N\delta(x-1),$$

where $\alpha>-1$, $\beta>-1$, $M\geqslant 0$, and $N\geqslant 0$. As a limiting case, Koornwinder also found the generalized Laguerre polynomials $\{L_n^{\alpha,M}(x)\}_{n=0}^{\infty}$, which are orthogonal on $[0, \infty)$ relative to the Laguerre weight plus a point mass at $x=0$ defined by

$$w_L^{(\alpha, M)}(x) = \frac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x} + M\delta(x),$$

where $M \geqslant 0$ and $\alpha > -1$. Recently, J. Koekoek and R. Koekoek found special types of infinite-order differential equations (see the Eqs. (4.17) and (4.18)), which have the sequences $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$ [16] and $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ [17] as eigenfunctions. These differential equations are, in general, of infinite order; in fact, they are of finite order only when α or β is a non-negative integer or $M=N=0$; see also Zhedanov [38], who found necessary conditions for $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ to satisfy a finite order differential equation (1.1).

In a series of papers [9–12], Grünbaum *et al.* extended Bochner's work further using Darboux transformations. In particular, they found in [12] a tenth order differential equation (1.1) having generalized Laguerre polynomials as solutions, which are orthogonal on $[0, \infty)$ relative to

$$e^{-x} + r_1\delta(x) - r_2\delta'(x).$$

Orthogonalizing moment functionals of all known *OPS*'s satisfying differential equation (1.1) are classical moment functionals plus point mass(es) at the end points of interval of the orthogonality of the corresponding classical moment functionals (see [12, 16, 17, 19, 32]). In this respect, Magnus [34] conjectured that if an *OPS* $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation (1.1), then $\{Q_n(x)\}_{n=0}^{\infty}$ must be orthogonal with respect to a classical weight function $\omega(x)$ plus point masses at the end points of the support of $\omega(x)$. We support Magnus' conjecture by showing that if a moment functional τ of the form

$$\tau = \sigma + v,$$

where σ is a classical moment functional and v $(\neq 0)$ is a distribution with finite support, has an *OPS* $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying the differential equation (1.1), then σ must be a moment functional for Jacobi or Laguerre or twisted Jacobi polynomials and *supp*(v) contains at most two points which are determined by σ (see Theorem 4.2). In particular, we find necessary and sufficient conditions for an *OPS* relative to $\tau := \sigma + M\delta(x-a) + N\delta(x-b)$, where σ is a classical moment functional, to satisfy the differential equation (1.1) (see Theorem 4.9), which completely explains the phenomenon of infinite order differential equations (see Eqs. (4.17) and (4.18)), which have $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ and $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$ as eigenfunctions. Finally, we give a new example of an *OPS* satisfying the equation (1.1) together with some related conjectures.

2. PRELIMINARIES

We let $\mathcal{P}$ be the space of all real polynomials in the single variable x and denote the degree of $\pi(x) \in \mathcal{P}$ by $deg(\pi)$, with the convention that $deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $deg(\phi_n) = n$ for each $n \in \mathbb{N}_0$. Note that a PS forms a basis for the vector space $\mathcal{P}$.

We call a linear functional σ: $\mathcal{P} \to \mathbb{R}$ a *moment functional* and, in distributional style, we denote its action on a polynomial $\pi(x)$ by $\langle \sigma, \pi \rangle$. For a moment functional σ, the numbers

$$\sigma_n := \langle \sigma, x^n \rangle \qquad (n \in \mathbb{N}_0)$$

are called the *moments* of σ. We say that a moment functional σ is *quasi-definite* (respectively, *positive-definite*) if its moments $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \qquad (\text{respectively, } \Delta_n(\sigma) > 0)$$

for every $n \in \mathbb{N}_0$. Any PS $\{\phi(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a non-zero constant multiple), called a canonical moment functional of $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle \sigma, \phi_0 \rangle \neq 0 \qquad \text{and} \qquad \langle \sigma, \phi_n \rangle = 0, \qquad n \geqslant 1.$$

DEFINITION 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial system (OPS) (respectively, a positive-definite OPS) if there is a moment functional σ satisfying

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn} \qquad (m, n \in \mathbb{N}_0), \tag{2.1}$$

where $\{K_n\}$ are non-zero (respectively, positive) real constants and δ_{mn} is the Kronecker delta function. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

It is immediate from (2.1) that, for any OPS $\{P_n(x)\}_{n=0}^{\infty}$, its orthogonalizing moment functional σ must be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Moreover, it is well known (see Chapter 1 in [4]) that a moment functional σ is quasi-definite if and only if there is an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ; furthermore, each $P_n(x)$ is uniquely determined up to a non-zero constant multiple.

Due to the representation theorems for the moment problem by Boas [2] and Duran [5], any moment functional σ has an integral representation of the form

$$\langle \sigma, \pi \rangle = \int_{-\infty}^{\infty} \pi(x)\, d\mu(x) = \int_{-\infty}^{\infty} \pi(x)\, \phi(x)\, dx \qquad (\pi \in \mathscr{P}),$$

where μ is a finite, signed Borel measure on $\mathbb{R}$ and $\phi(x)$ is a smooth, rapidly decaying function in the Schwartz space $\mathscr{S}(\mathbb{R})$. Hence, for any *OPS* $\{P_n(x)\}_{n=0}^{\infty}$, there is a distribution $w(x)$ relative to which $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal. In this case, we call $w(x)$ a distributional orthogonalizing weight for $\{P_n(x)\}_{n=0}^{\infty}$.

For a moment functional σ and a polynomial $\pi(x)$, we let σ' (the derivative of σ) and $\pi\sigma$ (the left multiplication of σ by $\pi(x)$) be the moment functionals defined by

$$\langle \sigma', \phi \rangle := -\langle \sigma, \phi' \rangle,$$

and

$$\langle \pi\sigma, \phi \rangle := \langle \sigma, \pi\phi \rangle$$

for $\phi(x) \in \mathscr{P}$. The following result is immediate from these definitions.

LEMMA 2.1 [29]. *For a moment functional σ and a polynomial $\pi(x)$, we have:*

(i) *Leibniz' rule:* $(\pi(x)\,\sigma)' = \pi'(x)\,\sigma + \pi(x)\,\sigma'$;

(ii) $\sigma' = 0$ *if and only if* $\sigma = 0$.

If σ is quasi-definite, then

(iii) $\pi(x)\,\sigma = 0$ *if and only if* $\pi(x) = 0$.

DEFINITION 2.2 [35]. A moment functional σ is semi-classical if:

(i) σ is quasi-definite, and

(ii) there exist polynomials $\phi(x)$ and $\psi(x)$ such that $(\phi, \psi) \neq (0, 0)$ and

$$(\phi\sigma)' - \psi\sigma = 0. \tag{2.2}$$

The quasi-definiteness of σ implies $\deg(\phi) \geqslant 0$ and $\deg(\psi) \geqslant 1$.

For any semi-classical moment functional σ, we call

$$s := \min\{\max(\deg(\phi) - 2, \deg(\psi) - 1)\}$$

the *class number* of σ, where the minimum is taken over all pairs of $(\phi, \psi) \neq (0, 0)$ of polynomials satisfying (2.2).

LEMMA 2.2 [36]. *Let σ be a semi-classical moment functional with class s satisfying $(\phi(x)\,\sigma)' = \psi(x)\,\sigma$. If $s = \max(\deg(\phi)-2, \deg(\psi)-1)$ and if σ also satisfies $(\phi_1(x)\,\sigma)' = \psi_1(x)\,\sigma$, then $\phi(x)$ divides $\phi_1(x)$.*

An *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ is called a semi-classical *OPS* (*SCOPS*) (of class s) if its canonical moment functional σ is semi-classical (with class number s). In particular, a *SCOPS* (respectively, semi-classical moment functional) of class 0 is called a classical *OPS* (respectively, a classical moment functional).

It is well known (see [27]) that an *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ is a classical *OPS* if and only if there are polynomials $A(x)$ and $B(x)$, independent of degree n, with $0 \leqslant \deg(A) \leqslant 2$ and $\deg(B) = 1$ such that

$$A(x)\,P_n''(x) + B(x)\,P_n'(x) = \left(\tfrac{1}{2}n(n-1)\,A''(x) + nB'(x)\right) P_n(x) \qquad (n \in \mathbb{N}). \tag{2.3}$$

Moreover, up to a real linear change of variable, there are only six classical *OPS*'s ([28]):

(i) Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ satisfying

$$(1-x^2)\,y''(x) + [(\beta-\alpha) - (\alpha+\beta+2)\,x]\,y'(x) = -n(n+\alpha+\beta+1)\,y(x)$$

$(-\alpha,\ -\beta$ and $-(\alpha+\beta+1) \notin \mathbb{N} = \{1, 2, ...\})$;

(ii) Bessel polynomials $\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfying

$$x^2 y''(x) + (\alpha x + 2)\,y'(x) = n(n+\alpha-1)\,y(x) \qquad (-\alpha+1 \notin \mathbb{N});$$

(iii) Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfying

$$xy''(x) + (\alpha+1-x)\,y'(x) = -ny(x) \qquad (-\alpha \notin \mathbb{N});$$

(iv) Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ satisfying

$$y''(x) - 2xy'(x) = -2ny(x);$$

(v) twisted Jacobi polynomials $\{\check{P}_n^{(d,e)}(x)\}_{n=0}^{\infty}$ satisfying

$$(x^2+1)\,y''(x) + (dx+e)\,y'(x) = n(n+d-1)\,y(x) \qquad (1-d \notin \mathbb{N});$$

(vi) twisted Hermite polynomials $\{\check{H}_n(x)\}_{n=0}^{\infty}$ satisfying

$$y''(x) + 2xy'(x) = 2ny(x).$$

Among the six classical *OPS*'s above, only the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ with α and $\beta > -1$, the Laguerre polynomials $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$ with $\alpha > -1$ and the Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ are positive-definite *OPS*'s. We denote the orthogonalizing moment functionals of the Jacobi, Bessel, Laguerre, Hermite, twisted Jacobi, and twisted Hermite polynomials by, respectively,

$$\sigma_J^{(\alpha,\beta)}, \qquad \sigma_B^{(\alpha)}, \qquad \sigma_L^{(\alpha)}, \qquad \sigma_H, \qquad \sigma_{\check{J}}^{(d,e)}, \qquad \text{and} \qquad \sigma_{\check{H}}.$$

Later, we will make use of the following simple observation: if the differential equation (2.3) has a classical *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ of solutions, then $B(x_0) \neq 0$ for any complex number x_0 where $A(x_0) = 0$.

3. BOCHNER–KRALL *OPS*'S

We call an *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ a *Bochner–Krall OPS* (*BKOPS*) of order N ($\geqslant 1$) (and write $\{P_n\} \in BKS(N)$; see [8]) if $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a differential equation (1.1) of order N but does not satisfy any differential equation (1.1) of order $< N$. Necessary and sufficient conditions for an *OPS* to be a *BKOPS* were found first by Krall [21], of which another simpler proof can be found in [29].

PROPOSITION 3.1 (see [20, 21, 29, 33]). *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an OPS relative to σ. Then the following statements are equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is a BKOPS satisfying the differential equation* (1.1);

(ii) *The moments $\{\sigma_n\}_{n=0}^{\infty}$ of σ satisfy $r := [\frac{N+1}{2}]$ recurrence relations*

$$S_k(m) := \sum_{i=2k+1}^{N} \sum_{j=0}^{i} \binom{i-k-1}{k} P(m-2k-1, i-2k-1)\, \ell_{i,i-j}\sigma_{m-j} = 0$$

for $k = 0, 1, \ldots, r-1$ and $m = 2k+1, 2k+2, \ldots$, where $P(n,k) = n(n-1)(n-2) \cdots (n-k+1)$;

(iii) *σ satisfies $r := [\frac{N+1}{2}]$ functional equations:*

$$R_k(\sigma) := \sum_{i=0}^{N-2k-1} (-1)^i \binom{i+k}{k} (\ell_{2k+i+1}\sigma)^{(i)} = 0 \qquad (3.1)$$

$$(k = 0, 1, \ldots, r-1);$$

(iv) $\sigma L_N[\cdot]$ *is symmetric on polynomials in the sense that*

$$\langle L_n(\phi)\,\sigma, \psi\rangle = \langle L_N(\psi)\,\sigma, \phi\rangle \qquad (\phi \text{ and } \psi \in \mathscr{P}).$$

Furthermore, if any of the above equivalent conditions holds, then $N = 2r$ *must be even.*

Proof. See Theorem 2.4 in [29]. ▮

Moreover, if the differential equation (1.1) has an *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ as eigenfunctions, then the differential operator $L_N[\cdot]$ must be Lagrangian symmetrizable (see [31]). However, in the case of Sobolev orthogonality, this result is not necessarily the case (see [7, 14]).

The equivalence of the statements (i) and (ii) was first shown by H. L. Krall [19] and the equivalence of (ii) and (iv) was established by Kwon *et al.* [29]. We call the r functional equations in (3.1) the *moment equations* for the differential equation (1.1). In particular, any *BKOPS* is a *SCOPS* and so $\deg(\ell_{2r-1}) \geqslant 1$ since

$$R_{r-1}[\sigma] = r(\ell_{2r}\sigma)' - \ell_{2r-1}\sigma = 0.$$

PROPOSITION 3.2. *If the differential equation* (1.1) *has an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *as solutions, then the moment equations* $R_k(\sigma) = 0$ $(0 \leqslant k \leqslant r-1)$ *have a unique non-trivial solution* σ*, up to a constant multiple, and* σ *must be quasi-definite.*

Proof. See Theorem 3.4 in [26]. ▮

By iteration, any *BKOPS* of order $2r$ satisfies differential equations of order $2r$, $4r$, ... However, we now show that for a *BKOPS* $\{P_n(x)\}_{n=0}^{\infty}$ of order $2r$, the $2r$th-order differential equation $L_{2r}[y] = \lambda_n y$ of the type (1.1) having $\{P_n(x)\}_{n=0}^{\infty}$ as solutions is unique up to a non-zero constant multiple.

PROPOSITION 3.3. *If the PS* $\{P_n(x)\}_{n=0}^{\infty}$ *satisfies the two differential equations*

$$L_N[y] = \sum_{i=1}^{N} \ell_i(x)\, y^{(i)}(x) = \lambda_n\, y(x) \qquad \text{with} \quad \ell_N \not\equiv 0$$

and

$$\tilde{L}_M[y] = \sum_{i=1}^{N} m_i(x)\, y^{(i)}(x) = \mu_n\, y(x) \qquad \text{with} \quad m_M \not\equiv 0$$

then $\ell_N^M(x) = C m_M^N(x)$ *for some constant* $C \neq 0$. *Thus for any BKOPS* $\{P_n(x)\}_{n=0}^{\infty}$ *of order* $2r$, *there is a unique (up to a non-zero constant multiple)* $2r$*th-order differential equation having* $\{P_n(x)\}_{n=0}^{\infty}$ *as solutions.*

Proof. The *PS* $\{P_n(x)\}_{n=0}^{\infty}$ also satisfies $(L_N \tilde{L}_M - \tilde{L}_M L_N)[P_n](x) = 0$ $(n \in \mathbb{N}_0)$ so that $L_N \tilde{L}_M = \tilde{L}_M L_N$. Now, with $D^j = d^j/dx^j$ for any $j \in \mathbb{N}$, we see that

$$\begin{aligned} L_N \tilde{L}_M[\,\cdot\,] &= \ell_N m_M D^{N+M} + (N\ell_N m_M' + \ell_{N-1} m_M + \ell_N m_{M-1}) \\ &\quad \times D^{N+M-1} + \cdots, \\ \tilde{L}_M L_N[\,\cdot\,] &= m_M \ell_N D^{N+M} + (M m_M \ell_N' + m_{M-1}\ell_N + m_M \ell_{N-1}) \\ &\quad \times D^{N+M-1} + \cdots; \end{aligned}$$

hence, $N\ell_N(x)\, m_M'(x) = M m_M \ell_N'(x)$. Consequently,

$$\frac{d}{dx}\left(\frac{m_M^N(x)}{\ell_N^M(x)}\right) = \ell_N^{M-1}(x)\, m_M^{N-1}(x)\, \frac{M m_M \ell_N'(x) - N\ell_N(x)\, m_M'(x)}{\ell_N^{2M}(x)} = 0.$$

Hence $\ell_N^M(x) = C m_M^N(x)$ for some constant $C \neq 0$. Now, the second claim follows immediately from the first. ∎

4. POINT MASS PERTURBATIONS OF CLASSICAL MOMENT FUNCTIONALS

Orthogonalizing moment functionals of all known *BKOPS*'s have at least one important point in common: they are one or two point mass perturbations of classical moment functionals. In this respect, A. Magnus [34] conjectured that $\mathscr{B} \subset \mathscr{K}$, where $\mathscr{B}$ is the class of *BKOPS*'s and $\mathscr{K}$ is the class of Koornwinder polynomials [18]; that is, *BKOPS*'s are *OPS*'s which are orthogonal relative to classical moment functionals plus point mass(es) at the end points of the interval of orthogonality. Conversely, we consider the problem: When is an *OPS* in the Koornwinder class a *BKOPS*? More general than Magnus' conjecture, we first consider a point mass perturbation $\tau := \sigma + \nu$ of a classical moment functional σ at an arbitrary number of points in the complex field $\mathbb{C}$, where

$$\nu = \sum_{k=1}^{m} \sum_{j=0}^{m_k} c_{k,j}\, \delta^{(j)}(x - x_k) \tag{4.1}$$

is a distribution with finite support $\{x_k\}_{k=1}^m$ in $\mathbb{C}$ and $c_{k,j} \in \mathbb{C}$. We also assume that

$$v = \bar{v} := \sum_{k=1}^{m} \sum_{j=0}^{m_k} \bar{c}_{k,j} \delta^{(j)}(x - \bar{x}_k)$$

so that v defines a real moment functional.

LEMMA 4.1. *Let σ be a quasi-definite moment functional. If for some polynomial $\pi(x)$, $\pi(x)\,\sigma = v$, where v is as in (4.1), then $\pi(x) \equiv 0$ and $v \equiv 0$.*

Proof. Let $\phi(x) = \prod_{k=1}^{m} (x - x_k)^{m_k+1}$. Then $\phi(x)\, v = 0$ so that $\phi(x)\, \pi(x)\, \sigma = \phi(x)\, v = 0$. Hence, by Lemma 2.1, $\phi(x)\, \pi(x) \equiv 0$ so that $\pi(x) \equiv 0$ and $v \equiv 0$. ∎

For the remainder of this paper, we shall assume that σ is a classical moment functional satisfying

$$(A(x)\,\sigma)' = B(x)\,\sigma,$$

where $0 \leqslant \deg(A) \leqslant 2$ and $\deg(B) = 1$. Then, by Proposition 3.1 with $N = 2$, the *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ satisfies the second-order differential equation

$$A(x)\, y''(x) + B(x)\, y'(x) = \lambda_n\, y(x).$$

Without loss of generality, we shall also assume that $\{P_n(x)\}_{n=0}^{\infty}$ is the monic classical *OPS* relative to σ. We are now in position to state one of our main results.

THEOREM 4.2. *Let $\tau := \sigma + v$ be a point mass perturbation of σ with v $(\neq 0)$ as in (4.1). If τ is also quasi-definite and gives rise to a BKOPS $\{Q_n(x)\}_{n=0}^{\infty}$ or order $\leqslant 2r$ satisfying*

$$L_{2r}[y](x) = \sum_{i=1}^{2r} \ell_i(x)\, y^{(i)}(x) = \lambda_n\, y(x), \tag{4.2}$$

then:

(i) *$supp(v) \subseteq \{x \in \mathbb{C} \mid A(x) = 0\}$ so that $m \leqslant 2$;*

(ii) *$A(x)$ divides $\ell_{2r}(x)$ and*

$$r\ell_{2r}(x)(B(x) - A'(x)) = A(x)(\ell_{2r-1}(x) - r\ell_{2r}'(x)); \tag{4.3}$$

(iii) *$R_{r-1}[\sigma] = 0$;*

(iv) *if* $x_0 \in supp(v)$ *is a zero of order* q $(\geqslant 1)$ *of* $\ell_{2r}(x)$, *then* x_0 *is a zero of order* $q-1$ *of* $\ell_{2r-1}(x)$;

(v) *the moment functional* σ *must be* $\sigma_J^{(\alpha,\beta)}$ *or* $\sigma_L^{(\alpha)}$ *or* $\sigma_{\check{J}}^{(d,e)}$; *furthermore,*

(a) *if* $\sigma = \sigma_J^{(\alpha,\beta)}$ *and* $1 \in supp(v)$ *(respectively,* $-1 \in supp(v)$*), then* α *(respectively,* β*) must be a non-negative integer;*

(b) *if* $\sigma = \sigma_L^{(\alpha)}$, *then* α *must be a non-negative integer;*

(c) *if* $\sigma = \sigma_{\check{J}}^{(d,e)}$, *then* $e = 0$ *and* $d = 2k$ *for some integer* $k \geqslant 1$.

In particular, Theorem 4.2 shows that in order to obtain a *BKOPS* by adding point masses to a classical moment functional σ, we can add only one or two mass points, which must be roots of $A(x)$.

In order to prove Theorem 4.2, we need the following facts for classical moment functionals, which are of interest in their own right.

LEMMA 4.3. *For any classical moment functional* σ *and any* $x_0 \in \mathbb{C}$, *we have*

$$\lim_{x \to x_0} (x - x_0) \frac{B(x) - A'(x)}{A(x)} \neq -1, -2, \ldots, \tag{4.4}$$

and

$$A^n(x)\, \sigma^{(n)} = \phi_n(x)\, \sigma \qquad (n \in \mathbb{N}_0), \tag{4.5}$$

where $\phi_n(x)$ *is a polynomial of degree* $\leqslant n$.

Proof. If $A(x_0) \neq 0$, then the left hand side of (4.4) is clearly equal to 0. If $A(x_0) = 0$, then (4.4) can be proved, case by case, for each of the six classical moment functionals. For $n = 0$, (4.5) holds with $\phi_0(x) = 1$. Assume that for an integer $\ell \geqslant 0$ there are polynomials $\phi_0(x), \phi_1(x), \ldots, \phi_\ell(x)$ with $\deg(\phi_i) \leqslant i$ $(0 \leqslant i \leqslant \ell)$ for which (4.5) holds for $n = 0, 1, \ldots, \ell$. Then

$$A^{\ell+1}(x)\, \sigma^{(\ell)} = A(x)\, \phi_\ell(x)\, \sigma. \tag{4.6}$$

Differentiating both sides of (4.6) gives

$$(\ell+1)\, A'(x)\, A^\ell(x)\, \sigma^{(\ell)} + A^{\ell+1}(x)\, \sigma^{(\ell+1)} = (\phi'_\ell(x)\, A(x) + \phi_\ell(x)\, B(x))\, \sigma,$$

so that, using $A^\ell(x)\, \sigma^{(\ell)} = \phi_\ell(x)\, \sigma$,

$$\begin{aligned} A^{\ell+1}(x)\, \sigma^{(\ell+1)} &= \{\phi'_\ell(x)\, A(x) + \phi_\ell(x)(B(x) - (\ell+1)\, A'(x))\}\, \sigma \\ &= \phi_{\ell+1}(x)\, \sigma, \end{aligned}$$

where $\phi_{\ell+1}(x) = \phi'_\ell(x)\, A(x) + \phi_\ell(x)(B(x) - (\ell+1)\, A'(x))$ is of degree $\leqslant \ell+1$. This completes the proof. ∎

Note that, for each $n \geqslant 0$, the polynomial $\phi_n(x)$ in (4.5) satisfies

$$\phi_{n+1}(x) = A(x)\, \phi'_n(x) + (B(x) - (n+1)\, A'(x))\, \phi_n(x) \qquad (n \in \mathbb{N}_0),$$

so that if $A(x_0) = 0$, then

$$\phi_{n+1}(x_0) = (B(x_0) - (n+1)\, A'(x_0))\, \phi_n(x_0) \qquad (n \in \mathbb{N}_0).$$

In particular, for the Jacobi, Bessel, Laguerre, and twisted Jacobi polynomials, routine calculations show that

$$\begin{cases} \phi_n(1) \neq 0 \ (n \in \mathbb{N}_0) & \text{if } \sigma = \sigma_J^{(\alpha,\beta)} \text{ and } \alpha \notin \mathbb{N}_0; \\ \phi_n(-1) \neq 0 \ (n \in \mathbb{N}_0) & \text{if } \sigma = \sigma_J^{(\alpha,\beta)} \text{ and } \beta \notin \mathbb{N}_0; \\ \phi_n(0) \neq 0 (n \in \mathbb{N}_0) & \text{if } \sigma = \sigma_B^{(\alpha)}; \\ \phi_n(0) \neq 0 (n \in \mathbb{N}_0) & \text{if } \sigma = \sigma_L^{(\alpha)} \text{ and } \alpha \notin \mathbb{N}_0; \\ \phi_n(\pm i) \neq 0 (n \in \mathbb{N}_0) & \text{if } \sigma = \sigma_{\check{J}}^{(d,e)} \text{ and } e \neq 0 \text{ or } \frac{1}{2}(d-2) \notin \mathbb{N}_0. \end{cases} \tag{4.7}$$

Proposition 4.4. *Assume that σ also satisfies $(\phi(x)\,\sigma)' = \psi(x)\,\sigma$ for some polynomials $\phi(x) \not\equiv 0$ and $\psi(x)$. Let $x_0 \in \mathbb{C}$ be a zero of order m $(\geqslant 1)$ of $\phi(x)$. Then:*

(i) *$A(x)$ divides $\phi(x)$;*

(ii) *if either $\sigma \neq \sigma_B^{(\alpha)}$ or $\sigma = \sigma_B^{(\alpha)}$ and $A(x_0) \neq 0$, then x_0 is a zero of order $m-1$ of $\psi(x)$;*

(iii) *if $\sigma = \sigma_B^{(\alpha)}$ and $A(x_0) = 0$, then x_0 is a zero of order $m-2$ of $\psi(x)$.*

Proof. Part (i) follows from Lemma 2.2. Since $(A(x)\,\sigma)' = B(x)\,\sigma$ and $(\phi(x)\,\sigma)' = \psi(x)\,\sigma$, we have by Lemma 2.2

$$\frac{B(x) - A'(x)}{A(x)} = \frac{\psi(x) - \phi'(x)}{\phi(x)}. \tag{4.8}$$

Write $\phi(x) = (x - x_0)^m\, \tilde{\phi}(x)$, where $\tilde{\phi}(x_0) \neq 0$. Then, by (4.8), we have

$$\psi(x) = (x - x_0)^m \frac{B(x) - A'(x)}{A(x)}\, \tilde{\phi}(x) + (x - x_0)^m\, \tilde{\phi}'(x) + m(x - x_0)^{m-1}\, \tilde{\phi}(x). \tag{4.9)}$$

If $m=1$, then we have from (4.9) and Lemma 4.3 that

$$\psi(x_0)=\tilde{\phi}(x_0)\left(\lim_{x\to x_0}(x-x_0)\frac{B(x)-A'(x)}{A(x)}+1\right)\neq 0.$$

If $m\geqslant 2$, then $\psi(x)=(x-x_0)^{m-2}\,\tilde{\psi}(x)$, where

$$\tilde{\psi}(x)=(x-x_0)^2\left(\frac{B(x)-A'(x)}{A(x)}\,\tilde{\phi}(x)+\tilde{\phi}'(x)\right)+m(x-x_0)\,\tilde{\phi}(x)$$

is real analytic. Hence x_0 is a zero of order at least $m-2$ of $\psi(x)$. If either $\sigma\neq\sigma_B^{(\alpha)}$ or $\sigma=\sigma_B^{(\alpha)}$ and $A(x_0)\neq 0$, then x_0 is a zero of $A(x)$ of order at most 1. Hence $\psi(x)=(x-x_0)^{m-1}\,(\tilde{\psi}(x)/(x-x_0))$ and, from Lemma 4.3,

$$\lim_{x\to x_0}\frac{\tilde{\psi}(x)}{x-x_0}=\lim_{x\to x_0}\tilde{\phi}(x)\left((x-x_0)\frac{B(x)-A'(x)}{A(x)}+m\right)\neq 0$$

so that x_0 is a zero of order $m-1$ of $\psi(x)$. If $\sigma=\sigma_B^{(\alpha)}$ and $A(x_0)=0$, then $A'(x_0)=0$ and $B(x_0)-A'(x_0)\neq 0$; hence

$$\tilde{\psi}(x_0)=\lim_{x\to x_0}(x-x_0)^2\frac{B(x)-A'(x)}{A(x)}\,\tilde{\phi}(x)=\frac{2(B(x_0)-A'(x_0))}{A''(x_0)}\,\tilde{\phi}(x_0)\neq 0;$$

that is, x_0 is a zero of order $m-2$ of $\psi(x)$. ∎

Proposition 4.5. *If there are polynomials $\pi_i(x)$ $(0\leqslant i\leqslant n)$ such that*

$$\nu:=\sum_{k=0}^{n}(\pi_k(x)\,\sigma)^{(k)}$$

is a distribution with finite support, then either $\nu=0$ or $A(x)=0$ for all $x\in supp(\nu)$. Consequently, supp(ν) contains at most two points. Moreover,

(i) *if $\sigma=\sigma_J^{(\alpha,\beta)}$ and $1\in supp(\nu)$ (respectively, $-1\in supp(\nu)$), then α (respectively, β) must be a non-negative integer;*

(ii) *if $\sigma=\sigma_L^{(\alpha)}$ and $0\in supp(\nu)$, then α must be a non-negative integer;*

(iii) *if $\sigma=\sigma_B^{(\alpha)}$ or σ_H or $\sigma_{\check{H}}$, then $\nu=0$;*

(iv) *if $\sigma=\sigma_{\check{J}}^{(d,e)}$ and $\nu\neq 0$, then $e=0$ and $d=2k$ for some integer $k\geqslant 1$.*

Proof. Assume $supp(\nu)=\{x_k\}_{k=1}^{m}$ and

$$\nu=\sum_{k=0}^{n}(\pi_k(x)\,\sigma)^{(k)}=\sum_{k=1}^{m}\sum_{j=0}^{m_k}c_{k,j}\delta^{(j)}(x-x_k)\neq 0.$$

Assume $A(x_1) \neq 0$. Then for the polynomial $\pi(x) := A^n(x) \prod_{k=2}^{m} (x-x_k)^{m_k+1}$, we have by (4.5),

$$\pi(x)\, v = \pi(x) \sum_{k=0}^{n} (\pi_k(x)\, \sigma)^{(k)} = \psi(x)\, \sigma,$$

for some polynomial $\psi(x)$. On the other hand, we also have

$$\begin{aligned}\pi(x)\, v &= \pi(x) \sum_{k=1}^{m} \sum_{j=0}^{m_k} c_{k,j} \delta^{(j)}(x-x_k) = \sum_{j=0}^{m_1} c_{1,j} \pi(x)\, \delta^{(j)}(x-x_1) \\ &= \sum_{j=0}^{m_1} c_{1,j} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \pi^{(i)}(x_i)\, \delta^{(j-i)}(x-x_1) \\ &= \sum_{i=0}^{m_1} \sum_{j=i}^{m_1} c_{1,j} (-1) \binom{j}{j-i} \pi^{(j-i)}(x_1)\, \delta^{(i)}(x-x_1).\end{aligned}$$

Hence, by Lemma 4.1, $\psi(x) = 0$ and

$$\sum_{j=i}^{m_1} c_{1,j} (-1)^{j-i} \binom{j}{j-i} \pi^{(j-i)}(x_1) = 0 \qquad (0 \leqslant i \leqslant m_1)$$

so that $c_{1,m_1} = c_{1,m_1-1} = \cdots = c_{1,0} = 0$ since $\pi(x_1) \neq 0$. Then $x_1 \notin supp(v)$, which is a contradiction. This proves the first assertion.

Assume that x_0 is a zero of $A(x)$ such that $\phi_n(x_0) \neq 0$ for each $n \in \mathbb{N}_0$ for any polynomial $\phi_n(x)$ in (4.5). Then we claim that $x_0 \notin supp(v)$. For $n = 0$, $\pi_0(x) \equiv 0$ and $v \equiv 0$ by Lemma 4.1. Hence $x_0 \notin supp(v)$. We assume that the claim holds for $n = 0, 1, 2, ..., \ell$. Let $v = \sum_{k=0}^{\ell+1} (\pi_k(x)\, \sigma)^{(k)}$ and $\pi_{\ell+1}(x) \not\equiv 0$. Then, by the first assertion, $supp(v) \subseteq \{x \in C \mid A(x) = 0\}$. Let $\pi_{\ell+1}(x) = q(x)\, A(x) + r(x)$, where $\deg(r) < \deg(A)$. Choose an integer $\tilde{m} \geqslant 0$ so that $A^{\ell+\tilde{m}+1}(x)\, v = 0$. Then

$$\begin{aligned}0 &= A^{\ell+\tilde{m}+1}(x)\, v = A^{\ell+\tilde{m}+1}(x) \left((\pi_{\ell+1}(x)\, \sigma)^{(\ell+1)} + \sum_{k=0}^{\ell} (\pi_k(x)\, \sigma)^{(k)} \right) \\ &= A^{\ell+\tilde{m}+1}(x) \left\{ \pi_{\ell+1}(x)\, \sigma^{(l+1)} + \sum_{k=0}^{\ell} \left(\binom{\ell+1}{k} \pi_{\ell+1}^{(\ell-k+1)}(x)\, \sigma^{(k)} \right.\right. \\ &\quad \left.\left. + (\pi_k(x)\, \sigma)^{(k)} \right) \right\} \\ &= A^{\tilde{m}}(x) (\pi_{\ell+1}(x)\, \phi_{\ell+1}(x) + A(x)\, \pi(x))\, \sigma\end{aligned}$$

for some polynomial $\pi(x)$ by Lemma 4.3. Hence, by Lemma 2.1(iii),

$$\pi_{\ell+1}(x)\,\phi_{\ell+1}(x)+A(x)\,\pi(x)=0;$$

that is,

$$A(x)(q(x)\,\phi_{\ell+1}(x)+\pi(x))=-r(x)\,\phi_{\ell+1}(x). \tag{4.10}$$

Since $A(x_0)=0$ and $\phi_{\ell+1}(x_0)\neq 0$, we see that $r(x_0)=0$; hence either $r(x)\equiv 0$ or $\deg(r)=1$. If $r(x)\equiv 0$, then

$$\begin{aligned} v=\sum_{k=0}^{\ell+1}(\pi_k(x)\,\sigma)^{(k)}&=(q(x)\,A(x)\,\sigma)^{(\ell+1)}+\sum_{k=0}^{\ell}(\pi_k(x)\,\sigma)^{(k)}\\ &=(q'(x)\,A(x)\,\sigma)^{(\ell)}+(q(x)\,B(x)\,\sigma)^{(\ell)}+\sum_{k=0}^{\ell}(\pi_k(x)\,\sigma)^{(k)} \end{aligned}$$

so that $x_0\notin supp(v)$ by our induction hypothesis. If $\deg(r)=1$, then $A(x)=r(x)\,s(x)$ where $\deg(s)=1$. Since $\phi_{\ell+1}(x_0)\neq 0$, we see that $s(x_0)\neq 0$ by (4.10). Then

$$\begin{aligned} s(x)\,v=s(x)\sum_{k=0}^{\ell+1}(\pi_k(x)\,\sigma)^{(k)}&=s(x)\left\{(\pi_{\ell+1}(x)\,\sigma)^{(\ell+1)}+\sum_{k=0}^{\ell}(\pi_k(x)\,\sigma)^{(k)}\right\}\\ &=s(x)\{(A(x)\,q(x)\,\sigma)^{(\ell+1)}+r(x)\,\sigma^{(\ell+1)}+(\ell+1)\,r'(x)\,\sigma^{(\ell)}\\ &\quad+\sum_{k=0}^{\ell}(\pi_k(x)\,\sigma)^{(k)}\Big\}\\ &=s(x)\{((q'(x)\,A(x)+q(x)\,B(x)+(\ell+1)\,r'(x))\,\sigma)^{(\ell)}\\ &\quad+\sum_{k=0}^{\ell}(\pi_k(x)\,\sigma)^{(k)}\Big\}+A(x)\,\sigma^{(\ell+1)}. \end{aligned}$$

Since

$$\begin{aligned} A(x)\,\sigma^{(\ell+1)}&=(A(x)\,\sigma)^{(\ell+1)}-(\ell+1)\,A'(x)\,\sigma^{(\ell)}-\binom{\ell+1}{2}A''(x)\,\sigma^{(\ell-1)}\\ &=(B(x)\,\sigma)^{(\ell)}-(\ell+1)\,A'(x)\,\sigma^{(\ell)}-\binom{\ell+1}{2}A''(x)\,\sigma^{(\ell-1)}, \end{aligned}$$

we have

$$s(x)\,v=\sum_{k=0}^{\ell}(\tilde{\pi}_k(x)\,\sigma)^{(k)}$$

for some polynomials $\tilde{\pi}_k(x)$. By our induction hypothesis, $x_0 \notin supp(s(x)\,v)$ so that $x_0 \notin supp(v)$ since $s(x_0) \neq 0$. Hence, by (4.7), the proof is complete. ▮

PROPOSITION 4.6. *If $\tau := \sigma + v$ satisfies $(\phi\tau)' = \psi\tau$ for some polynomials $\phi(x)$ and $\psi(x)$, then $A(x)$ divides $\phi(x)$ and*

$$\phi(x)(B(x) - A'(x)) = A(x)(\psi(x) - \phi'(x)). \tag{4.11}$$

Proof. Since

$$(\phi\tau)' - \psi\tau = \phi\sigma' + (\phi' - \psi)\,\sigma + (\phi v)' - \psi v = 0,$$

we see that

$$\phi A\sigma' + A(\phi' - \psi)\,\sigma = [\phi(B - A') + A(\phi' - \psi)]\,\sigma = A\psi v - A(\phi v)';$$

hence (4.11) follows from Lemma 4.1. Let $x_0 \in \mathbb{C}$ be any zero of $A(x)$. If $B(x_0) - A'(x_0) \neq 0$, then $\phi(x_0) = 0$ by (4.11). If $B(x_0) - A'(x_0) = 0$, then we consider the following two cases separately: $B(x) - A'(x) \not\equiv 0$ or $B(x) - A'(x) \equiv 0$. Assume first that $B(x_0) - A'(x) = 0$ and $B(x) - A'(x) \not\equiv 0$. Then $B(x) - A'(x) = a(x - x_0)$ for some $a \neq 0$. Set

$$\phi(x) = q(x)\,A(x) + r(x) \qquad (\deg(r) < \deg(A)).$$

Then, by (4.11), we have

$$ar(x)(x - x_0) = A(x)[q(x)(A'(x) - B(x)) + \psi(x) - \phi'(x)]$$

so that $q(x)(A'(x) - B(x)) + \psi(x) - \phi'(x) = b$, for some constant b. Set $A(x) = (x - x_0)\,\tilde{A}(x)$. Then $r(x) = c\tilde{A}(x)$, where $c = b/a$ and so $(A\sigma)' = B\sigma$ becomes $(x - x_0)(\tilde{A}\sigma' - a\sigma) = 0$. Hence

$$\tilde{A}(x)\,\sigma' = a\sigma + \lambda\delta(x - x_0),$$

for some constant $\lambda \neq 0$. Hence

$$(\phi\tau)' - \psi\tau = (q'A + qB + r' - \psi + ac)\,\sigma + c\lambda\delta(x - x_0) + (\phi v)' - \psi v = 0.$$

Then, by Lemma 4.1, $q'(x)\,A(x) + q(x)\,B(x) + r'(x) - \psi(x) + ac = 0$ and

$$c\lambda\delta(x - x_0) + (\phi v)' - \psi v = 0. \tag{4.12}$$

Let

$$v = \sum_{k=1}^{m} \sum_{j=0}^{m_k} c_{k,j}\delta^{(j)}(x - x_k) \qquad (c_{k,m_k} \neq 0).$$

If $x_0 \notin \{x_k\}_{k=1}^m$, then $c\lambda = 0$ by (4.12) so that $c = 0$ and $r(x) = 0$. Hence $\phi(x_k) = \phi(x_0) = 0$. If $x_0 \in \{x_k\}_{k=1}^m$, then $x_0 = x_k$ for some k so that

$$c\lambda\delta(x-x_k) + \phi(x) \sum_{j=0}^{m_k} c_{k,j}\delta^{(j+1)}(x-x_k)$$

$$+ (\phi'(x) - \psi(x)) \sum_{j=0}^{m_k} c_{k,j}\delta^{(j)}(x-x_k)$$

$$= \phi(x_k)\, c_{k,m_k}\delta^{(m_k+1)}(x-x_k) + \sum_{j=0}^{m_k} d_{k,j}\delta^{(j)}(x-x_k) = 0$$

by (4.12). Hence $\phi(x_k) = \phi(x_0) = 0$. Finally, assume that $B(x) - A'(x) \equiv 0$. Then either $\sigma = \sigma_J^{(0,0)}$ or $\sigma = \sigma_{\check{J}}^{(2,0)}$. If $\sigma = \sigma_J^{(0,0)}$, then $(1-x^2)\,\sigma' = 0$ and hence

$$\sigma' = \delta(x+1) - \delta(x-1)$$

(assuming $\langle \sigma, 1\rangle = 2$). Hence $\psi(x) - \phi'(x) \equiv 0$ by (4.11) and so

$$(\phi\tau)' - \psi\tau = \phi\sigma' + (\phi v)' - \psi v$$
$$= \phi(-1)\,\delta(x+1) - \phi(1)\,\delta(x-1) + (\phi v)' - \psi v = 0.$$

Then $\phi(-1) = \phi(1) = 0$ by the same reasoning as above. If $\sigma = \sigma_{\check{J}}^{(2,0)}$, then $(1+x^2)\,\sigma' = 0$ so that (again, assuming $\langle \sigma, 1\rangle = 2$),

$$\sigma' = i\delta(x-i) - i\delta(x+1),$$

where $i = \sqrt{-1}$. Similarly, as for $\sigma_J^{(0,0)}$, we have $\phi(-i) = \phi(i) = 0$. In all cases, we have shown that $\phi(x_0) = 0$ for any root x_0 of $A(x)$. Hence $A(x)$ divides $\phi(x)$. ∎

We now give a proof of Theorem 4.2:

Proof. Let $v_1(x)$ be the restriction of $v(x)$ on $\{x \in \mathbb{C} \mid A(x) = 0\}$. Then we can decompose $v(x)$ as

$$v(x) = v_1(x) + v_2(x).$$

By Proposition 3.1, τ satisfies the r moment equations

$$0 = R_k(\tau) = \sum_{i=0}^{2r-2k-1} (-1)^i \binom{i+k}{k} (\ell_{2k+i+1}(x)\,\sigma)^{(i)}$$
$$+ \sum_{i=0}^{2r-2k-1} (-1)^i \binom{i+k}{k} (\ell_{2k+i+1}(x)\,v_1)^{(i)}$$
$$+ \sum_{i=0}^{2r-2k-1} (-1)^i \binom{i+k}{k} (\ell_{2k+i+1}(x)\,v_2)^{(i)}$$

for $k=0, 1, \ldots, r-1$. Let $\tilde{\tau} := \sigma + v_1$. Then, by the first part of Proposition 4.5, the last sum $R_k(v_2)$ must be zero and so $\tilde{\tau}$ also satisfies the r moment equations

$$R_k(\tilde{\tau}) = 0$$

for $k=0, 1, \ldots, r-1$. By Proposition 3.2, $\tau = \tilde{\tau}$; that is, $v_2 = 0$. Hence, part (i) of the theorem is proved. Part (v) follows from the second part of Proposition 4.5, while parts (ii) and (iv) follow from Proposition 4.6 and Proposition 4.4, respectively. Finally, to prove (iii), set $\ell_{2r}(x) = q(x)\, A(x)$. By (4.3),

$$\ell_{2r-1}(x) = r(q(x)\, B(x) + q'(x)\, A(x))$$

so that $R_{r-1}(\sigma) = r(q(x)\, A(x)\, \sigma)' - \ell_{2r-1}(x)\, \sigma = 0$. ∎

As a special case of Theorem 4.2, we have:

THEOREM 4.7. *Let $\tau = \sigma + \lambda \delta^{(m)}(x-a) + \mu \delta^{(n)}(x-b)$, where $a \neq b$ and m, $n \in \mathbb{N}_0$. Assume that τ is quasi-definite and gives rise to a BKOPS $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying the differential equation* (4.2). *If $\lambda \neq 0$, then $\ell_{2r}(x)$ (respectively, $\ell_{2r-1}(x)$) vanishes of order at least $m+2$ (respectively, at least $m+1$) at a. If $\mu \neq 0$, then $\ell_{2r}(x)$ (respectively, $\ell_{2r-1}(x)$) vanishes of order at least $n+2$ (respectively, at least $n+1$) at b.*

Proof. We assume $\lambda \neq 0$. The case for $\mu \neq 0$ can be proved in a similar way. By Theorem 4.2, $R_{r-1}[\tau] = R_{r-1}[\sigma] = 0$ so that

$$R_{r-1}[\lambda \delta^{(m)}(x-a) + \mu \delta^{(n)}(x-b)] = 0.$$

Hence

$$\begin{aligned} 0 &= R_{r-1}[\delta^{(m)}(x-a)] \\ &= r\ell_{2r}(x)\, \delta^{(m+1)}(x-a) + (r\ell'_{2r}(x) - \ell_{2r-1}(x))\, \delta^{(m)}(x-a) \\ &= r\ell_{2r}(a)\, \delta^{(m+1)}(x-a) + (-1)^{m+1}\, \ell_{2r-1}^{(m)}(a)\, \delta(x-a) - \sum_{j=1}^{m} (-1)^{m+j} \\ &\quad \times \left\{ r\binom{m}{j-1} \ell_{2r}^{(m+1-j)}(a) + \binom{m}{j} \ell_{2r-1}^{(m-j)}(a) \right\} \delta^{(j)}(x-a). \end{aligned}$$

Hence $\ell_{2r}(a) = \ell_{2r-1}^{(m)}(a) = 0$ and

$$r\binom{m}{j-1}\ell_{2r}^{(m+1-j)}(a)+\binom{m}{j}\ell_{2r-1}^{(m-j)}(a)=0 \qquad (1\leqslant j\leqslant m).$$

That is, $\ell_{2r}(a) = \ell_{2r-1}^{(m)}(a) = 0$ and

$$r(m-j+1)\,\ell_{2r}^{(j)}(a)+j\ell_{2r-1}^{(j-1)}(a)=0 \qquad (1\leqslant j\leqslant m). \tag{4.13}$$

On the other hand, by Theorem 4.2, $A(a) = \ell_{2r}(a) = 0$. Let q ($\geqslant 1$) be the order of zero of $x = a$ for $\ell_{2r}(x)$. Then, by Theorem 4.2, $\ell_{2r-1}(x)$ has $x = a$ as a zero of order $q-1$. Hence

$$\ell_{2r}(x)=(x-a)^q\,\tilde{\ell}_{2r}(x), \qquad \tilde{\ell}_{2r}(a)=\frac{1}{q!}\,\ell_{2r}^{(q)}(a)\neq 0;$$

$$\ell_{2r-1}(x)=(x-a)^{q-1}\,\tilde{\ell}_{2r-1}(x), \qquad \tilde{\ell}_{2r-1}(a)=\frac{1}{(q-1)!}\,\ell_{2r-1}^{(q-1)}(a)\neq 0.$$

Then, by (4.3),

$$\frac{B(x)-A'(x)}{A(x)}=\frac{\tilde{\ell}_{2r-1}(x)-rq\tilde{\ell}_{2r}(x)}{r(x-a)\,\tilde{\ell}_{2r}(x)}-\frac{\tilde{\ell}'_{2r}(x)}{\tilde{\ell}_{2r}(x)},$$

so that

$$\lim_{x\to a}(x-a)\frac{B(x)-A'(x)}{A(x)}=\frac{q}{r}\,\frac{\ell_{2r-1}^{(q-1)}(a)-r\ell_{2r}^{(q)}(a)}{\ell_{2r}^{(q)}(a)}. \tag{4.14}$$

If $1\leqslant q\leqslant m+1$, then by (4.13) and (4.14)

$$\lim_{x\to a}(x-a)\frac{B(x)-A'(x)}{A(x)}=-(m+1),$$

which contradicts Lemma 4.3. Hence $q\geqslant m+2$ and the conclusion follows from Theorem 4.2 (iv). ∎

In particular, consider

$$\tau := \sigma + M\delta(x-a)+N\delta(x-b), \tag{4.15}$$

where $a \neq b$ and σ is a classical moment functional satisfying $(A(x)\sigma)' = B(x)\sigma$ with $0 \leqslant \deg(A) \leqslant 2$ and $\deg(B) = 1$. Then, the moment functional τ in (4.15) is quasi-definite if and only if

$$(4.16) \qquad d_n := \left| \begin{pmatrix} 1 + MK_n(a, a) & NK_n(a, b) \\ MK_n(a, b) & 1 + NK_n(b, b) \end{pmatrix} \right| \neq 0 \qquad (n \in \mathbb{N}_0),$$

where $K_n(x, y) := \sum_{k=0}^{n} (P_k(x) P_k(y)/\langle \sigma, P_k^2 \rangle)$ is the kernel polynomial of the monic classical *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ (see Theorem 3.1 in [30]). In this case, τ is also a semi-classical moment functional (see Theorem 5.2 in [30]).

From Theorem 4.2 and Theorem 4.7, we have

COROLLARY 4.8. *If τ in (4.15), with $M \neq 0$, gives rise to a BKOPS $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying the differential equation (4.2), then σ must be $\sigma_J^{(\alpha, \beta)}$ or $\sigma_L^{(\alpha)}$ or $\sigma_{\check{J}}^{(d, 0)}$, where α or β is a non-negative integer, $d = 2k$ for some integer $k \geqslant 1$, and:*

(i) $A(a) = NA(b) = 0$;

(ii) $\ell_{2r}(a) = \ell'_{2r}(a) = \ell_{2r-1}(a) = N\ell_{2r}(b) = N\ell'_{2r}(b) = N\ell_{2r-1}(b) = 0$;

(iii) $\ell_{2r}(x) = A(x)\,\phi(x)$ *and* $\ell_{2r-1}(x) = r(\phi(x)\,B(x) + \phi'(x)\,A(x))$ *for some polynomial* $\phi(x)$ *with* $\phi(a) = N\phi(b) = 0$;

(iv) $R_{r-1}[\sigma] = 0$.

In particular, Corollary 4.8 explains why the Bessel type orthogonal polynomials found by Hendriksen [13] cannot satisfy a finite order differential equation of the form (1.1).

We now consider in detail the three cases for the moment functionals $\sigma = \sigma_J^{(\alpha, \beta)}$, $\sigma_L^{(\alpha)}$ and $\sigma_{\check{J}}^{(d, e)}$. Let $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$, $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$, and $\{\check{P}_n^{d, e, M, \bar{M}}(x)\}_{n=0}^{\infty}$ be the Jacobi type, the Laguerre type, and the twisted Jacobi type polynomials, which are orthogonal relative, respectively, to the weight distributions

$$\tau_J = \sigma_J^{(\alpha, \beta)} + M\delta(x+1) + N\delta(x-1) \qquad (M, N \in \mathbb{R}),$$
$$\tau_L = \sigma_L^{(\alpha)} + M\delta(x) \qquad (M \in \mathbb{R}),$$
$$\tau_{\check{J}} = \sigma_{\check{J}}^{(d, e)} + M\delta(x+i) + \bar{M}\delta(x-i) \qquad (M \in \mathbb{C}).$$

We normalize $\sigma_J^{(\alpha, \beta)}$, $\sigma_L^{(\alpha)}$, and $\sigma_{\check{J}}^{(d, e)}$ so that $\langle \sigma_J^{(\alpha, \beta)}, 1 \rangle = \langle \sigma_L^{(\alpha)}, 1 \rangle = \langle \sigma_{\check{J}}^{(d, e)}, 1 \rangle = 1$.

For any *PS* $\{Q_n(x)\}_{n=0}^{\infty}$, there are infinitely many differential equations of infinite order with polynomial coefficients,

$$L[y](x)=\sum_{i=1}^{\infty} \ell_i(x)\, y^{(i)}(x)=\lambda_n\, y(x)$$

which have $\{Q_n(x)\}_{n=0}^{\infty}$ as eigenfunctions. Here $\ell_i(x)=\sum_{j=0}^{i} \ell_{ij}x^j$ and

$$\lambda_n=\ell_{11}n+\ell_{22}n(n-1)+\cdots+\ell_{nn}n!.$$

To be precise, Krall and Sheffer [24] (see also [1, 15]) showed that for any sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$, with $\lambda_0=0$ and $\lambda_m\neq\lambda_n$ for $m\neq n$, there is a unique sequence of polynomials $\{\ell_i(x)\}_{i=1}^{\infty}$ such that $L[Q_n]=\lambda_n Q_n$ for each $n\in\mathbb{N}_0$. In this respect, J. Koekoek and R. Koekoek (see [17]) found infinite-order differential equations for the generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ and the generalized Laguerre polynomials $\{L_n^{\alpha,M}(x)\}_{n=0}^{\infty}$ (see [16]). To summarize their work, they showed that the generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ satisfy a unique differential equation of the form

$$\begin{aligned}(4.17)\qquad 0=M\sum_{i=0}^{\infty} a_i(x)\, y^{(i)}+N\sum_{i=0}^{\infty} b_i(x)\, y^{(i)}+MN\sum_{i=0}^{\infty} c_i(x)\, y^{(i)}\\ +(1-x^2)\, y''+(\beta-\alpha-(\alpha+\beta+2)\, x)\, y'+n(n+\alpha+\beta+1)\, y,\end{aligned}$$

where $a_i(x)$, $b_i(x)$, $c_i(x)$ are polynomials of degree $\leqslant i$ and are independent of n for $i\geqslant 1$. Moreover, the order $O_J(\alpha,\beta)$ of this differential operator is given by

$$O_J(\alpha,\beta)=\begin{cases}\infty & \text{if } M>0 \text{ and } \beta\notin N_0 \text{ or } N>0 \text{ and } \alpha\notin N_0\\ 2 & \text{if } M=N=0\\ 2\alpha+4 & \text{if } M=0,\ N>0, \text{ and } \alpha\in N_0\\ 2\beta+4 & \text{if } M>0,\ N=0, \text{ and } \beta\in N_0\\ 2\alpha+2\beta+6 & \text{if } M>0,\ N>0, \text{ and } \alpha \text{ and } \beta\in N_0.\end{cases}$$

Here, in the latter four cases, the leading coefficient is given by

$$\begin{cases}1-x^2 & \text{if } M=N=0\\ \dfrac{-1}{(\beta+1)_{\alpha+1}}\dfrac{(x^2-1)^{\alpha+2}}{(\alpha+2)!} & \text{if } M=0,\ N>0, \text{ and } \alpha\in N_0\\ \dfrac{-1}{(\alpha+1)_{\beta+1}}\dfrac{(x^2-1)^{\beta+2}}{(\beta+2)!} & \text{if } M>0,\ N=0, \text{ and } \beta\in N_0\\ \dfrac{-(\alpha+\beta+2)}{(\alpha+1)(\beta+1)}\dfrac{(x^2-1)^{\alpha+\beta+3}}{(\alpha+\beta+1)!\,(\alpha+\beta+3)!} & \text{if } M>0,\ N>0, \text{ and } \alpha,\beta\in N_0.\end{cases}$$

The generalized Laguerre polynomials $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$ satisfy a unique differential equation of the form

$$M \sum_{i=0}^{\infty} a_i(x)\, y^{(i)} + xy'' + (\alpha+1-x)\, y' + ny = 0, \tag{4.18}$$

where each $a_i(x)$ is a polynomial of degree $\leqslant i$ and is independent of n for $i \geqslant 1$. Moreover, the order $O_L(\alpha)$ of the differential operator in (4.18) is

$$O_L(\alpha) = \begin{cases} \infty & \text{if } \; M > 0 \; \text{ and } \; \alpha \notin N_0 \\ 2 & \text{if } \; M = 0 \\ 2\alpha + 4 & \text{if } \; M > 0 \; \text{ and } \; \alpha \in N_0, \end{cases}$$

and in the latter two cases, the leading coefficient is

$$\begin{cases} x & \text{if } \; M = 0 \\ \dfrac{(-1)^{\alpha+1}}{(\alpha+2)!}\, x^{\alpha+2} & \text{if } \; M > 0 \; \text{ and } \; \alpha \in N_0. \end{cases}$$

Following Koornwinder [18], who first introduced the generalized Jacobi polynomials and the generalized Laguerre polynomials, J. Koekoek and R. Koekoek assumed $\alpha, \beta > -1$ and $M, N \geqslant 0$ in [16, 17]; under these assumptions, $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ and $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$ are positive-definite *OPS*'s. However, we can relax these restrictions on the parameters α, β, M, N by the condition (4.16) to obtain quasi-definite *OPS*'s $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$ and $\{L_n^{\alpha, M}(x)\}_{n=0}^{\infty}$, which still satisfy the differential equation (4.17) and (4.18), respectively.

THEOREM 4.9. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be an OPS relative to the moment functional τ defined in* (4.15). *Then $\{Q_n(x)\}_{n=0}^{\infty}$ is a BKOPS if and only if*

(i) $M = N = 0$ *or*

(ii) $\tau = \tau_J$ *with* $\alpha \in \mathbb{N}_0$ *when* $N \neq 0$ *and* $\beta \in \mathbb{N}_0$ *when* $M \neq 0$ *or*

(iii) $\tau = \tau_L$ *with* $\alpha \in \mathbb{N}_0$ *or*

(iv) $\tau = \tau_{\tilde{J}}$ *with* $d = 2k$ *(k a positive integer) and* $e = 0$.

Proof. The necessity follows by Theorem 4.2(v). The sufficiency of the condition (i) is trivial. The sufficiency of conditions (ii) and (iii) follows from the fact that the differential equations (4.17) and (4.18) are of finite order under the given conditions. Finally, consider the *OPS* $\{Q_n(x)\}_{n=0}^{\infty} := \{\check{P}_n^{2k,0,M,\bar{M}}(x)\}_{n=0}^{\infty}$, where k is a positive integer. Set

$$P_n(x) := i^{-n} Q_n(ix) \qquad (i = \sqrt{-1}\,;\, n \in \mathbb{N}_0).$$

Then $\{P_n(x)\}_{n=0}^{\infty}$ is an *OPS* relative to the moment functional

$$\tilde{\sigma} + M\delta(x+1) + \bar{M}\delta(x-1),$$

where $\tilde{\sigma}$ is the moment functional defined by $\langle \tilde{\sigma}, \pi(x)\rangle = \langle \sigma_J^{(2k,0)}, \pi(-ix)\rangle$ for any polynomial $\pi(x)$. Since $\sigma := \sigma_J^{(2k,0)}$ satisfies $((1+x^2)\,\sigma)' = 2kx\sigma$, $\tilde{\sigma}$ satisfies the functional equation $((1-x^2)\,\tilde{\sigma})' = -2kx\tilde{\sigma}$ so that $\tilde{\sigma} = \sigma_J^{(k-1,k-1)}$ is the Jacobi moment functional. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation (4.17) (see Eqs. (14)–(16) in [17]):

$$0 = M \sum_{j=0}^{2k+2} a_j(x)\, y^{(j)} + \bar{M} \sum_{j=0}^{2k+2} b_j(x)\, y^{(j)} + |M|^2 \sum_{j=0}^{4k+2} c_j(x)\, y^{(j)}$$
$$+ (1-x^2)\, y'' - 2kxy' + n(n+2k-1)\, y.$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty}$ is a *BKOPS* satisfying the differential equation

(4.19)
$$0 = M \sum_{j=0}^{2k+2} i^j a_j(-ix)\, y^{(j)} + \bar{M} \sum_{j=0}^{2k+2} i^j b_j(-ix)\, y^{(j)}$$
$$+ |M|^2 \sum_{j=0}^{4k+2} i^j c_j(-ix)\, y^{(j)}$$

The differential equation (4.19) has real polynomials as coefficients since we have (see Eqs. (7)–(11) in [17]):

$$a_j(x) = \sum_{\ell=0}^{j-1} (-1)^{\ell+1}\, a_{j,\ell}(x+1)^{\ell+1} \qquad (j \geqslant 1)$$

$$b_j(x) = (-1)^j \sum_{\ell=0}^{j-1} a_{j,\ell}(x-1)^{\ell+1} \qquad (j \geqslant 1)$$

$$c_1(x) = 0 \qquad \text{and} \qquad c_j(x) = c_j^{(1)}(x) + c_j^{(2)}(x) \qquad (j \geqslant 2),$$

where

$$c_j^{(1)}(x) = (x^2-1) \sum_{\ell=0}^{j-2} (-1)^{\ell+1}\, c_{j,\ell}(x+1)^{\ell+1}$$

$$c_j^{(2)}(x) = (-1)^j\,(x^2-1) \sum_{\ell=0}^{j-2} c_{j,\ell}(x-1)^{\ell+1},$$

and where $a_{j,\ell}$ and $c_{j,\ell}$ are real constants independent of M. ∎

Note that Theorem 4.9, together with Theorem 4.2, completely characterize *BKOPS*'s which are orthogonal relative to $\tau := \sigma + \nu$, where σ is a classical moment functional and ν is a distribution of order 0 with finite support.

EXAMPLE 4.1. In 1982, Littlejohn [32] found a *BKOPS* of order 6, called the Krall polynomials $\{K_n(x)\}_{n=0}^{\infty} = \{K_n(A, B; x)\}_{n=0}^{\infty}$, which are orthogonal relative to

$$\tau = \sigma_J^{(0,0)} + \frac{1}{A}\,\delta(x+1) + \frac{1}{B}\,\delta(x-1) \qquad (A, B \in \mathbb{R} \backslash \{0\})$$

and satisfy the sixth-order differential equation

$$\begin{aligned} L_6[y](x) = {} & (x^2-1)^3\, y^{(6)}(x) + 18x(x^2-1)^2\, y^{(5)}(x) \\ & + \{3(A+B+32)\,x^4 - 6(A+B+22)\,x^2 + 3(A+B+12)\}\, y^{(4)}(x) \\ & \times 24(A+B+7)(x^3-x)\, y^{(3)}(x) + \{(12AB+42(A+B)+72)\,x^2 \\ & + 12(B-A)\,x - (12AB+30(A+B)+72)\}\, y''(x) \\ & + 12\{(2AB+A+B)\,x + B - A\}\, y'(x) \\ = {} & \lambda_n\, y(x). \end{aligned}$$

Moreover, these polynomials are explicitly given by

$$\begin{aligned} K_n(x) = {} & \sum_{j=0}^{[n/2]} \frac{(-1)^j\,(2n-j)!\,(n^2-n+1+B+4j)\,x^{n-2j}}{2^{n+1}(n-j)!\,j!(n-2j)!} \\ & - \sum_{j=0}^{[n/2]} \frac{(-1)^j\,(2n-2j)!\,(A-B)^2\,x^{n-2j}}{2^{n+1}(n-j)!\,j!(n-2j)!\,(n^2+n+A+B)} \\ & + \sum_{j=0}^{[n-1/2]} \frac{(-1)^j\,(2n-2j-1)!\,(B-A)\,x^{n-2j-1}}{2^{n-1}(n-j-1)!\,j!(n-2j-1)!\,(n^2+n+A+B)} \end{aligned}$$

and satisfy the three-term recurrence relation

$$\begin{aligned} K_n(x) = {} & \frac{(2n-1)\,A(n)\,B(n-1)}{nB(n)\,A(n-1)}\,xK_{n-1}(x) \\ & + \frac{(2n-1)(2B-2A)\,C(n)\,B(n-1)}{nB(n)[A(n-1)]^2}\,K_{n-1}(x) \\ & - \frac{(n-1)\,B(n-2)[A(n)]^2}{nB(n)[A(n-1)]^2}\,K_{n-2}(x), \end{aligned} \tag{4.20}$$

where

$$A(n) = n^4 + (2A + 2B - 1)\, n^2 + 4AB$$
$$B(n) = n^2 + n + A + B$$
$$C(n) = -3n^4 + 6n^3 - (2A + 2B + 3)\, n^2 + 2(A + B)\, n + 4AB.$$

Define

$$\check{K}_n(x) = i^{-n} K_n(A + iB, A - iB; ix) \qquad (n \in \mathbb{N}_0,\ A, B \in \mathbb{R}).$$

Then, $\{\check{K}_n(x)\}_{n=0}^{\infty}$ is a real *PS* satisfying the three-term recurrence relation (4.20) where $K_n(x)$, $A(n)$, $B(n)$, and $C(n)$ are replaced by, respectively, $\check{K}_n(x)$ and

$$\check{A}(n) = n^4 + (4A - 1)\, n^2 + 4(A^2 + B^2)$$
$$\check{B}(n) = n^2 + n + 2A$$
$$\check{C}(n) = -3n^4 + 6n^3 - (4A + 3)\, n^2 + 4An + 4(A^2 + B^2).$$

Hence, if $\check{A}(n) \neq 0$ and $\check{B}(n) \neq 0$, $(n \in \mathbb{N}_0)$, then $\{\check{K}_n(x)\}_{n=0}^{\infty}$ is a *BKOPS* relative to

$$\sigma_{\check{J}}^{(2,0)} + \frac{1}{A + iB}\,\delta(x - i) + \frac{1}{A - iB}\,\delta(x + i),$$

and satisfies

$$\begin{aligned}\check{L}_6[y](x) = {} & (x^2+1)^3\, y^{(6)}(x) + 18x(x^2+1)^2\, y^{(5)}(x) \\ & + 6\{(A+16)\, x^4 + 2(A+11)\, x^2 + A + 6\}\, y^{(4)}(x) \\ & + 24(2A+7)(x^3+x)\, y^{(3)} \\ & + 12\{(A^2+B^2+7A+6)\, x^2 - 2Bx + A^2 + B^2 + 5A + 6\}\, y''(x) \\ & + 24\{(A^2+B^2+A)\, x - B\}\, y'(x) \\ = {} & \lambda_n\, y(x).\end{aligned}$$

Note that $\{\check{K}_n(x)\}_{n=0}^{\infty}$ is symmetric if and only if $B = 0$.

Finally we make some conjectures on the *BKOPS* class, which will improve Magnus' conjecture. Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a *BKOPS* relative to τ of order $2r \geqslant 4$ satisfying the differential equation (4.2). Then we conjecture:

(C-1) $\tau = \sigma + \nu$, where σ is a classical moment functional satisfying $(A(x)\,\sigma)' = B(x)\,\sigma$ with some polynomials $A(x)$ of degree $\leqslant 2$, $B(x)$ of degree 1 and ν is a distribution with its support at the zeros of $A(x)$;

(C-2) $\ell_{2r}(x) = A(x)^r$ and $\ell_{2r-1}(x) = rA(x)^{r-1}\,[(r-1)\,A'(x) + B(x)]$ (see (4.3)).

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